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On the Meaning of $E = mc^2$

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Abstract

Einstein's original derivation of the energy–mass relation is re-examined. It is shown that while his conclusion that gamma emission from an excited nucleus must accompany a reduction of the inertial (rest) mass of the nucleus is valid for a structureless particle, it is not necessarily valid when the complex structure of the nucleus is taken into account. In the latter case, in addition to the change in kinetic energy of the entire emitting nucleus that was considered in Einstein's analysis, one must also take account of the change in nuclear configuration energies, from the period before to the period after de-excitation. It is then concluded that the inertial mass of a body (i.e. its resistance to a change of state of motion) of a gamma-emitting nucleus could be exactly the same before and after emission if the internal configuration energy of the nucleus would be correspondingly altered in the process. An experimental test that utilizes the Mössbauer effect is suggested. Einstein's further conclusion that mass is a measure of the energy content of matter is questioned with reference to the *conceptual differences* between the inertial and energetic features of matter. It is concluded that $E = mc^2$ is not an identity (i.e. an 'if-and-only-if' relation) but it is rather an 'if-then' relation, with meaningful connotation only in the local domain, where the formalism of special relativity theory is a useful approximation for a generally relativistic formulation for theories of matter.

1. *Introduction*

It is sometimes useful in theoretical physics to re-examine the conceptual content of 'well-established' theories. For after the passage of several generations of scientists, if no experimental evidence appears to challenge the mathematical equations that were originally found to represent these theories, there is a (human!) tendency to lose sight of the original concept in all of its subtlety, holding onto the equations alone. The History of Science has taught us that continual re-examination of the *conceptual structure* of a theory in physics can indeed lead to new views that, in turn, can imply additional predictions of experimental effects, not sought or

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recognised before. With these reflections in mind, I should like to re-consider in this paper the perhaps most famous result of Einstein's relativity theory—the energy–mass relation $E = mc^2$.

2. *Einstein's Original Analysis*

In one of his earliest papers Einstein (1905) analysed the process in which a quantity of radioactive matter emits gamma radiation. To express the effect that was sought most simply, he described the decay process in the center-of-momentum system. Thus, starting with the assumption that the quantity of energy of gamma radiation $\frac{1}{2}E_\gamma$ is emitted at the angle θ with respect to the x-direction, and in the opposite direction, $\theta + \pi$, the quantity of energy $\frac{1}{2}E_\gamma$ is simultaneously emitted, the energy conservation law then asserts that

$$E_0 = E_1 + \tfrac{1}{2}E_\gamma + \tfrac{1}{2}E_\gamma \tag{2.1}$$

where E_0 and E_1 are the respective total matter energies, before and after the emission of the gamma radiation.

In the next step, Einstein described the same process of gamma emission from the frame of reference of an observer who is moving in the x-direction, with a velocity $\mathbf{v}$ with respect to the emitting nucleus. In the latter frame, the respective initial and final energies of the decaying nucleus are called E_0' and E_1'. According to the application of the Lorentz transformations to the description of the emitted gamma radiation in the observer's frame (i.e. applied to the variables of Maxwell's equations) the latter quantities of (oppositely directed) electromagnetic energy have the following forms in the moving frame of reference:

$$\tfrac{1}{2}E_\gamma \frac{1 - (v/c)\cos\theta}{(1 - (v/c)^2)^{1/2}} \quad \text{and} \quad \tfrac{1}{2}E_\gamma \frac{1 + (v/c)\cos\theta}{(1 - (v/c)^2)^{1/2}}$$

Thus, in the moving frame of reference the energy conservation law (2.1) takes the following form:

$$\begin{aligned} E_0' &= E_1' + \tfrac{1}{2}E_\gamma \frac{1 - (v/c)\cos\theta}{(1 - (v/c)^2)^{1/2}} + \tfrac{1}{2}E_\gamma \frac{1 + (v/c)\cos\theta}{(1 - (v/c)^2)^{1/2}} \\ &= E_1' + \frac{E_\gamma}{(1 - (v/c)^2)^{1/2}} \end{aligned} \tag{2.2}$$

Combining equations (2.1) and (2.2) then leads to the following:

$$(E_0' - E_0) - (E_1' - E_1) = E_\gamma((1 - (v/c)^2)^{-1/2} - 1) \tag{2.3}$$

With this result, Einstein argued as follows: The only physical difference between the quantities defined in the primed and unprimed coordinate frames is concerned with whether or not there is motion relative to the emitting nucleus. He then concluded that one must interpret the respective quantities of energy, $(E_0' - E_0)$ and $(E_1' - E_1)$, as the *kinetic energy* of the observed nucleus, before and after the gamma emission—a type of energy that arises strictly by virtue of the observer's motion relative to the gamma-decaying nucleus. Calling the respective kinetic energy values K_0 and K_1 for the relative motion before and after the emission of the gamma radiation, and taking v/c to be much less than unity (by experimental arrangement), and noting that in this limit the kinetic energy, according to special relativity theory, is approximately equal to the classical form, $\frac{1}{2}mv^2$, equation (2.3) takes the following approximate form:

$$K_0 - K_1 \equiv \delta(\tfrac{1}{2}mv^2) = \tfrac{1}{2}(v/c)^2 E_\gamma \tag{2.4}$$

The right-hand side of equation (2.4) is obtained by applying the binomial expansion to the right side of equation (2.3), with v/c small.

Noting now that, by experimental arrangement, the relative velocity of the moving observer does not change from the times before to the times after the emission of the gamma radiation, a change in kinetic energy in equation (2.4) can only mean a change in the inertial mass of the emitting nucleus. Thus, for sufficiently small velocities, $K_0 - K_1 \equiv \delta(\frac{1}{2}mv^2) = \frac{1}{2}(\delta m)v^2$. It then follows from equation (2.4) that

$$E_\gamma = (\delta m)\, c^2 \tag{2.5}$$

This result then led Einstein to the conclusion that

(A) '...if a body gives off energy in the form of radiation, its mass diminishes by E_γ/c^2.'

He then asserted that

> 'It is not impossible that with bodies whose energy-content is variable to a high degree (e.g. with radium salts) the theory may be successfully put to the test.'

and

(B) 'the mass of a body is a measure of its energy content.'

When Einstein's analysis is applied to the case of an unstable, structureless elementary particle, his result (A) is quite conclusive. And in the case of complex nuclei, there is, of course, abundant experimental evidence to support his energy–mass relation (2.5), such as the observations of mass defect in nuclear disintegration. Nevertheless, there seem to me to be two important criticisms (in view of our present-day knowledge) of Einstein's conclusions (A) *and* (B), as general assertions.

3. *Examination of Einstein's Conclusion (A)*

Einstein's assumptions that led to equation (2.5) are certainly valid when applied to a quantity of matter that has no structure, or when its structure can be neglected. But when we consider a quantity of elementary matter, such as a complex nucleus, decaying by gamma radiation, there are also changes in velocity-dependent potential energy terms, in addition to the kinetic energy of the whole nucleus that was considered by Einstein. These are terms on the left side of equation (2.3) that are involved in the mutual binding of the constituent nucleons, before and after gamma emission has taken place. In this case, the left side of equation (2.3) cannot be simply identified with the kinetic energy of the emitting nucleus, as a whole, as it is done in equation (2.4).

To be explicit, consider a simple model in which electrostatic forces contribute to the binding of the constituent nucleons in a configuration in the stationary (center-of-momentum) frame of reference, similar to the description of a crystalline solid.† In this case, the total intrinsic energy in the stationary frame of the nucleus is the electrostatic term

$$\int \sum_{i<k} [\rho(i)\,\phi(k)]_0\, d\mathbf{r}_i$$

where the summation is taken over all charged nucleons and (i,k) refer to the states of motion of these nucleons, as represented in the stationary frame.

In the moving frame of the observer of the decaying nucleus, this electrostatic energy term has the following Lorentz-transformed expression:

$$\int \sum_{i<k} [\rho'(i')\,\phi'(k') - \mathbf{j}'(i').\mathbf{A}'(k')]_0\, d\mathbf{r}_i' \equiv \int \sum_{i<k} [j^{\mu'}(i')\,A_\mu'(k')]_0\, d\mathbf{r}_i' \quad (3.1)$$

where the subscript '0' refers to the expectation values of the energy expressions, when the nucleus is in its initial, excited state (i.e. before emission). Similar terms appear with the subscript '1', referring to the nucleus in its final, de-excited state (after emission). The indices (i',k') denote the states of motion of the respective constituent nucleons, as described in the observer's frame of reference.

In the left side of equation (2.3), then, there are two parts—one that entails the kinetic energy of the nucleus as a whole, relative to a given observer, and the other that entails the changes in electromagnetic configuration energy relative to this observer. Denoting the former by K_0 and K_1, as before, equation (2.4) should now appear as follows:

$$(K_0 - K_1) + (I_0 - I_1) = \tfrac{1}{2}(v/c)^2 E_\gamma \quad (3.2)$$

where

$$I_\alpha = \left\{ \int \sum_{i<k} j^{\mu'}(i')\,A_\mu'(k')\, d\mathbf{r}_i' - \int \sum_{i<k} \rho(i)\,\phi(k)\, d\mathbf{r}_i \right\}_\alpha$$

and $\alpha = 0,\ 1$ denote the electromagnetic configuration energies in the respective excited and de-excited states of the nucleus.

† The main binding of a complex nucleus is, of course, due to the mutually acting nuclear forces. However, the conclusions of the discussion that follows is insensitive to the types of forces considered, except that they are specified to be velocity-dependent, in their general expression in an arbitrary frame of reference. The electromagnetic interactions that are involved in the binding of nuclear configurations in the excited and de-excited states are those which are pertinent to nuclear decay by gamma emission.

Since the latter are velocity-dependent binding energy contributions, and '1' refers to a state of lower energy than the state '0', it follows that $(I_0 - I_1)$ is a positive quantity. Further, since the velocity $\mathbf{v}$ of the observer of the decaying nucleus is constant, and the same before and after the gamma emission, it follows, as in Einstein's analysis, that the change in kinetic energy of the radioactive nucleus, as a whole, from the time preceding to the time following the gamma emission, can only be described in terms of a change of the inertial mass of the nucleus (not a velocity change). That is to say, as before when $v/c \ll 1$, it follows that $K_0 - K_1 = \frac{1}{2}(\delta m) v^2$, where δm is the change in inertia of the complex nucleus, as a whole. Equation (3.2) then takes the following form:

$$\tfrac{1}{2}(\delta m) v^2 + |I_0 - I_1| = \tfrac{1}{2}(v/c)^2 E_\gamma$$

or

$$(\delta m) c^2 + 2(c/v)^2 |I_0 - I_1| = E_\gamma > (\delta m) c^2 \qquad (3.3)$$

Thus the energy associated with a change in inertial mass of a decayed complex nucleus *plus* the change in configuration energy of the complex nucleus is equal to the energy that is carried off in gamma radiation. The latter is greater than the change in the rest energy of the nucleus if a change in the nuclear configuration energy occurs.

Of course, it is most likely that a part of the energy of de-excitation is converted from the rest energy of the whole nucleus and a part from the configuration energy. But what is the actual distribution? The answer to this question, from theory, would depend on a detailed knowledge of the eigenstates of the complex nucleus. To determine the distribution of energies experimentally would require an accurate comparison of the measured values of the nuclear mass, before and after it has emitted gamma radiation—as Einstein originally suggested (Einstein, 1905). However, I am not aware of any data on gamma-emitting nuclei, at the present time, that could definitely answer the question.

The results of this analysis would be significant in the interpretation of experimental results that are sensitive to the inertia of a nucleus, rather than to its energy content. For example, the sharp gamma spectrum observed in the Mössbauer experiment (Mössbauer, 1958a, b, 1959) entails the emission of gamma radiation from the constituent nucleus (say radioactive Ir^{191}) of a macroscopic crystal lattice, with its recoil distributed equally throughout the entire lattice, rather than in one nucleus. Now if the inertial mass of the emitting nucleus should actually change when the radiation is emitted, then the dynamics of the entire crystal would change. This is because the constants of the motion associated with the vibrational states of the crystal lattice are sensitive to the *inertial* properties of the constituent nuclei. On the other hand, if the gamma decay of a constituent Ir^{191} nucleus should entail a change in the configuration of its nuclear structure without any change in its rest energy, then the lattice dynamics should be the same before and after the de-excitation had occurred. The implication here is that a close analysis of the Mössbauer effect might be able to answer the question as to whether or not there is, in particular cases, inertial mass changes in a complex nucleus, when it should emit gamma radiation.

4. *Examination of Einstein's Conclusion* (*B*)

The preceding example is very much tied to Einstein's conclusion (B)—that the measure of the inertial mass of a body is identically a measure of its energy content. I do not believe that this is a valid logical conclusion if it is intended to infer that the mass–energy relation is an 'if-and-only-if' relation. This is because inertia and energy refer to *conceptually different* features of matter. On the one hand, inertia *per se* relates to the resistance to the change of state of motion of a body, as would be caused, in 'particle physics', by an external force acting on this body (i.e. a force rooted in *other matter*). In general relativity theory, with the incorporation of the Mach principle, the inertial mass of any quantity of matter is a measure of its dynamical coupling, as a component of a *closed system*, with all of the other matter components of that system. With this view, inertia is a *global property* of a material system.

On the other hand, energy *per se* is defined in terms of the solutions of a conservation law. This type of physical law follows, most fundamentally, from the invariance of the formal expression of the theory with respect to arbitrary, continuous changes in the time coordinate. But the latter refers only to 'local time'—*the time measure of a local observer*. In the global expression of the general form of relativity theory, it is well known that there are no conservation laws. Thus, unlike the inertial mass concept, 'energy' is an undefined concept in the global domain. The role of the energy concept in physics is that energy differences (rather than absolute quantities of energy) are among the theoretical predictions that are to be identified with the actual measured values for the properties of matter in any observer's local frame of reference. In general relativity theory, global tensors (i.e. generally covariant mathematical entities) necessarily appear, that relate only asymptotically to energy, *per se*. But these do not become energy until the asymptotic limit is taken, corresponding to the special relativistic (i.e. local) limit of the more general expressions (Sachs, 1969).

Thus we see that energy and inertial mass are conceptually different (though complementary) features of matter, according to relativity theory. But this does not mean that the formula $E = mc^2$ is a scientifically false statement! It only means that this is an equation rather than an equivalence. That is to say, in the language of formal logic, $E = mc^2$ is an 'if-then' rather than an 'if-and-only-if' relation. Thus, in principle, one must first derive the inertial mass of a quantity of matter from the global properties of a closed system. Once this is done, its locally asymptotic limit, m, must be determined and then *inserted into* the formula $E = mc^2$ in order to derive the corresponding rest energy that would be deduced from measurements.

But one cannot proceed generally in the opposite direction—that of deriving the *global* feature, inertia, from a given value of the *local* feature, energy. This would be to make the logically fallacious claim of being able to derive a unique universal from one (or any number of) particular(s).

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